Journal of Global Optimization (2005) 31: 421–436
DOI 10.1007/s10898-004-6874-2

Optimality Conditions for Vector Optimization Problems of a Difference of Convex Mappings

A. TAA
Département de Mathématiques, Faculté des Sciences et Techniques, B. P. 549, Marrakech, Morocco. (e-mail: taa@fstg-marrakech.ac.ma)

(Received 29 May 2002; revised 27 June 2003; accepted: 4 Janurary 2004.)

Abstract. In this paper, we study optimality conditions for vector optimization problems of a difference of convex mappings

$$(VP)\begin{cases} \mathbb{R}^p_+- & -\text{Minimize} f(x)-g(x), \\ & \text{subject to the constraints} \\ & x\in C, l(x)\in -Q, Ax=b \\ & \text{and } h(x)-k(x)\in -\mathbb{R}^m_+, \end{cases}$$

where $f:=(f_1,\dots,f_p)$, $g:=(g_1,\dots,g_p)h:=(h_1,\dots,h_m)$, $k:=(k_1,\dots,k_m)$, Q is a closed convex cone in a Banach space Z, l is a mapping Q-convex from a Banach space X into Z, A is a continuous linear operator from X into a Banach space $W, \mathbb{R}^p_+$ and $\mathbb{R}^m_+$ are respectively the nonnegative orthants of $\mathbb{R}^p$ and $\mathbb{R}^m$, C is a nonempty closed convex subset of X, $b\in W$, and the functions f_i, g_i, h_j and k_j are convex for $i=1,\dots,p$ and $j=1, ldots, m$. Necessary optimality conditions for (VP) are established in terms of Lagrange-Fritz-John multipliers. When the set of constraints for (VP) is convex and under the generalized Slater constraint qualification introduced in Jeyakumar and Wolkowicz [11] , we derive necessary optimality conditions in terms of Lagrange-Karush-Kuhn-Tucker multipliers which are also sufficient whenever the functions $g_i, i=1,\dots,p$ are polyhedrals. Our approach consists in using a special scalarization function. A necessary optimality condition for convex vector maximization problem is derived. Also an application to vector fractional mathematical programming is given. Our contribution extends the results obtained in scalar optimization by Hiriart-Urruty [9] and improve substantially the few results known in vector case (see for instance: [11], [12] and [14]).

Mathematics Subject Classifications (1991). Primary: 90C29; Secondary 49K30.

Key words: Vector optimization, Difference of convex mappings, Optimality conditions, Fractional mathematical programming, Lagrange multipliers.

1. Introduction

Vector optimization has drawn much attention for a long time, and many results have been obtained. For instance, Censor [3] gives optimality conditions for differentiable convex vector optimization by using the theorem of Dubovitskii-Milyutin. Yu [21] generalized Pareto-optimal solutions to nondominated cone solutions in the objective space of multiobjective programming problems.

Charnes et al. [5] studied the properties of nondominated solutions in decision spaces which are normed vector space, these authors further developed new approaches [6] and applied them to extensions of game theory. Li [13] developed new necessary as well as sufficient conditions for nondominated solutions to cone-quasiconvex multiobjective programming problems by assuming the quasiconvexity of the weighted sum of subgroups of objectives. Taa [20] studied optimality conditions in terms of Lagrange–Fritz–John and Lagrange–Karush–Kuhn–Tucker multipliers for nonsmooth and nonconvex vector mathematical programming with the existence of the Hadamarad directional derivatives of objective and constraint functions.

In this paper, we investigate optimality conditions in terms of Lagrange–Fritz–John and Lagrange–Karush–Kuhn–Tucker multipliers for vector optimization problems when, objective and constraints are defined by difference of convex mappings. Our approach consists in using a special scalarization function introduced in optimization by Hiriart–Urruty [8]. Our contribution extends the results obtained in scalar case by Hiriart-Urruty [9] and improve substantially the few results known in convex vector case, see for instance Minami [14], Kanniappan [12], and Jeyakumar and Wolkowicz [11].

Our paper is organized in this way. In Section 2, we will recall the results by Attouch and Brezis [2], and Swartz [19], and we will establish some preliminary results. Section 3 is devoted to necessary optimality conditions in terms of Lagrange–Fritz–John multipliers for (VP) (see Theorem 3.1). When the set of constraints is convex and under the generalized Slater constraint qualification introduced by Jeyakumar et al. [11], we establish the Lagrange–Karush–Kuhn–Tucker necessary optimality conditions for (VP) (see Corollary 3.2) which are also sufficient whenever the functions $g_i, i = 1, \dots, p$ are polyhedrals (see Corollary 3.3). A necessary optimality condition for convex vector maximization problems is derived. In section 4, we give an application to vector fractional mathematical programming.

2. Preliminary

Throughout this paper X, Z and W are Banach spaces whose topological dual spaces are X^*, Z^* and W^* respectively. Let $K \subset \mathbb{R}^p$ be a pointed ($K \cap -K = \{0\}$) closed convex cone with its interior $\mathrm{Int}K \neq \emptyset$. Let A be a nonempty subset of $\mathbb{R}^p$. The weak K-minimal set and the weak K-maximal set of A are defined by

$$W \cdot \mathrm{Min}_K A = \left\{ \overline{y} \in A : A \subset \overline{y} + \mathbb{R}^p \setminus -\mathrm{Int}K \right\}$$

and

$$W \cdot \mathrm{Max}_K A = \left\{ \overline{y} \in A : A \subset \overline{y} + \mathbb{R}^p \setminus \mathrm{Int}K \right\}$$

respectively. The polar cone K^o of K is defined as

$$K^o=\{y^*\in\mathbb{R}^p:\langle y^*,y\rangle\leqslant 0 \text{ for all } y\in K\},$$

where $\langle .,.\rangle$ is the dual pairs.

Given a mapping $\varphi:X\to\mathbb{R}^p$, the epigraph of φ is defined by

$$\operatorname{epi}(\varphi)=\{(x,y)\in X\times Y: y\in\varphi(x)+K\}.$$

Since convexity plays an important role in the following investigations, recall the concept of cone–convex mappings.

The mapping φ is said to be K-convex if for every $\alpha\in[0,1]$ and $x_1,x_2\in X$,

$$\alpha\varphi(x_1)+(1-\alpha)\varphi(x_2)\in\varphi(\alpha x_1+(1-\alpha)x_2)+K.$$

The problem considered in this paper can be formulated as follows

$$(VP)\begin{cases}\mathbb{R}^p_+- & \text{Minimize } f(x)-g(x),\\ & \text{subject to the constraints}\\ & x\in C, l(x)\in -Q, Ax=b\\ & \text{and } h(x)-k(x)\in-\mathbb{R}^m_+,\end{cases}$$

where $f:=(f_1,\dots,f_p), g:=(g_1,\dots,g_p):X\to\mathbb{R}^p, h:=(h_1,\dots,h_m), k:=(k_1,\dots,k_m):X\to\mathbb{R}^m, A:X\to W$ is a continuous linear operator, Q is a closed convex cone in Z with $\operatorname{Int}Q\neq\emptyset, l$ is a mapping Q-convex from X into Z, $\mathbb{R}^p_+$ and $\mathbb{R}^m_+$ are respectively the nonnegative orthants of $\mathbb{R}^p$ and $\mathbb{R}^m$, C is a nonempty closed convex subset of X, and the functions f_i, g_i, h_j and k_j are convex for $i=1,\dots,p$ and $j=1,\dots,m$. We let

$$D=\{x\in X: h(x)-k(x)\in-\mathbb{R}^m_+ \text{ and } l(x)\in -Q\} \tag{1}$$

and

$$E=\{x\in X: Ax=b\}. \tag{2}$$

We let

$$F=C\cap D\cap E \tag{3}$$

denote the feasible set of (VP). Consider the set

$$(f-g)(F):=\{f(x)-g(x): x\in F\}.$$

$\overline{x}$ is a local weak minimal solution of (VP) with respect to $\mathbb{R}^p_+$ if $\overline{x}\in F$ and if there exists a neighborhood V of $\overline{x}$ such that $f(\overline{x})-g(\overline{x})\in W\cdot\operatorname{Min}_{\mathbb{R}^p_+}((f-g)(F\cap V))$.

Consider the following maximization problem with respect to $\mathbb{R}^p_+$:

$$(\mathrm{VP}')\begin{cases}\mathbb{R}^p_+- & \text{Maximize}\, G(x,y),\\ & \text{subject to the constraints}\\ & x\in F\,\text{and}\, M(x,y)\in -\mathbb{R}^p_+,\end{cases}$$

where $G(x,y)=g(x)-y$ and $M(x,y)=f(x)-y$ for all $(x,y)\in X\times\mathbb{R}^p$.
Clearly both G and M are $\mathbb{R}^p_+$-convex on $X\times\mathbb{R}^p$. Put

$$F' := \left\{(x,y)\in X\times\mathbb{R}^p : x\in F\,\text{and}\, M(x,y)\in -\mathbb{R}^p_+\right\}.$$

$(\overline{x},\overline{y})$ is a local weak maximal solution of (VP′) with respect to $\mathbb{R}^p_+$ if $(\overline{x},\overline{y})\in F'$ and if there exists a neighborhood U of $(\overline{x},\overline{y})$ such that

$$G(\overline{x},\overline{y})\in W\cdot \mathrm{Max}_{\mathbb{R}^p_+} G(F'\cap U).$$

As in scalar case (see Hiriart-Urruty [9]), one easily check that if $\overline{x}$ is a local weak minimal solution of (VP) with respect to $\mathbb{R}^p_+$, then $(\overline{x}, f(\overline{x}))$ is a local weak maximal solution of (VP′) with respect to $\mathbb{R}^p_+$ and, conversely, if $(\overline{x},\overline{y})$ is a local weak maximal solution of (VP′) with respect to $\mathbb{R}^p_+$, then $\overline{y}=f(\overline{x})$ and $\overline{x}$ is a local weak minimal solution of (VP) with respect to $\mathbb{R}^p_+$. Hence, a vector minimization of a difference of convex mappings in X can be viewed as a convex maximization problem in $X\times\mathbb{R}^p$.

The next concept is introduced in Dauer and Saleh [7].

DEFINITION 2.1 [7]. Let Y be a Banach space and A be a nonempty subset of Y. A functional $\psi : A\to\mathbb{R}$ is called Y^+–increasing (respectively decreasing) on A, if for each $y_0\in A$

$$y\in(y_0+Y^+)\cap Y \,\text{implies}\, \psi(y)\geqslant (\text{respectively} \leqslant)\,\psi(y_0),$$

where Y^+ is a nonempty closed convex cone of Y.

For a subset S of a Banach space Y, we consider the function

$$\Delta_S(y)=\begin{cases} d(y,S) & \text{if}\, y\in(Y\setminus S)\\ -d(y,\mathbb{R}^p\setminus S) & \text{if}\, y\in S\end{cases}$$

where $d(y,S)=\inf\{\|u-y\| : u\in S\}$. This function is introduced in Hiriart–Urruty [8] (see also [10]), and used by Ciligot -Travain [4], and Amahroq and Taa [1].

Let us recall the following result of [19].

PROPOSITION 2.1 [19]. *Let Y be a Banach space and $S\subset Y$ be a closed convex cone with nonempty interior and $S\neq Y$. The function Δ_S is convex, positively homogeneous, 1-Lipschitzian, increasing on Y with respect to* *the order introduced by S. Moreover* $(Y \setminus S) = \{y \in Y : \Delta_S(y) > 0\}$, $IntS = \{y \in Y : \Delta_S(y) < 0\}$ *and the boundary of* $S : bd(S) = \{y \in Y : \Delta_S(y) = 0\}$.

It is easy to verify the following lemma.

LEMMA 2.1. *The function* $\Phi : \mathbb{R}^p \times Z \times \mathbb{R}^m \to \mathbb{R}$ *defined by*

$$\Phi(y, z, w) = \max(\Delta_{-\mathrm{Int}\mathbb{R}^p_+}(y), \Delta_{-Q}(z), \Delta_{-Int\mathbb{R}^m_+}(w))$$

is $(\mathbb{R}^p_+ \times Q \times \mathbb{R}^m_+)$*-increasing on* $\mathbb{R}^p \times Z \times \mathbb{R}^m$.

Given a convex function $\psi : X \to \mathbb{R} \cup \{+\infty\}$. The subdifferential $\partial\psi(\overline{x})$ of ψ at $\overline{x} \in \mathrm{dom}(\psi)$ is defined as

$$\partial\psi(\overline{x}) = \left\{x^* \in X^* : \psi(x) - \psi(\overline{x}) \geqslant < x^*, x - \overline{x} > \text{ for all} \, x \in X\right\},$$

where $\mathrm{dom}(\psi) = \{x \in X : \psi(x) < +\infty\}$.

Let B be a closed convex subset of X. The normal cone $N_B(\overline{x})$ of B at $\overline{x}$ is denoted

$$N_B(\overline{x}) = \left\{x^* \in X^* : 0 \geqslant < x^*, x - \overline{x} > \text{ for all} \, x \in B\right\}.$$

This cone can be also written as

$$N_B(\overline{x}) = \partial\delta_B(\overline{x}),$$

where δ_B is the indicator function of B (i.e., $\delta_B = 0$ if $x \in B$ and $\delta_B = +\infty$ if $x \notin B$). Properties of the subdifferential and the normal cone can be found in Rockafellar [17].

As a direct consequence of Proposition 2.1, one has the following result.

PROPOSITION 2.2 [4]. *Let* Y *be a Banach space and* $S \subset Y$ *be a nonempty closed convex cone with nonempty interior. Then for all* $y \in Y, 0 \notin \partial\Delta_S(y)$.

The following result has been proved by Attouch and Brezis [2] in the Banach space setting and by Rodrigues and Simons [18] in the case of the Frechet space.

THEOREM 2.1 [2]. *Assume that* $\psi_1, \psi_2 : X \to \mathbb{R} \cup \{+\infty\}$ *are convex, lower semicontinuous and proper and that* $\mathbb{R}^+(\mathrm{dom}(\psi_1) - \mathrm{dom}(\psi_2))$ *is a closed vector subspace of X. Then*

$$\partial(\psi_1 + \psi_2)(x) = \partial\psi_1(x) + \partial\psi_2(x).$$

3. Optimality Conditions

In this section, we preserve the notations given in the previous section and we give optimality conditions for (VP) in terms of Lagrange–Fritz–John and Lagrange–Karush–Kuhn–Tucker multipliers. Before stating the theorem which gives necessary optimality conditions for a local weak minimal solution of (VP), we introduce the following lemma.

LEMMA 3.1 *If $\overline{x}$ is a local weak minimal solution of* (VP) *with respect to $\mathbb{R}^p_+$, then for all $(x_1^*, \dots, x_p^*) \in \partial g_1(\overline{x}) \times \dots \times \partial g_p(\overline{x})$ and $(w_1^*, \dots, w_m^*) \in \partial k_1(\overline{x}) \times \dots \times \partial k_m(\overline{x})$, $\overline{x}$ solves the following unconstraint scalar optimization problem*

$$(SP)\begin{cases}\text{Minimize} & \max(\Delta_{-\mathrm{Int}\mathbb{R}^p_+}(H_1(x)), \Delta_{-Q}(l(x)), \Delta_{-\mathbb{R}^m_+}(H_2(x))) + \delta_{C\cap E}(x)\\ \text{subject to} & x \in X,\end{cases}$$

where $H_1(x) = f(x) - f(\overline{x}) - (\langle x_1^*, x - \overline{x}\rangle, \dots, \langle x_p^*, x - \overline{x}\rangle)$, $H_2(x) = h(x) - k(\overline{x}) - (\langle w_1^*, x - \overline{x}\rangle, \dots, \langle w_m^*, x - \overline{x}\rangle)$ and E is defined by relation (2).

Proof. Suppose the contrary. By convexity of the following function

$$x \to \max(\Delta_{-\mathrm{Int}\mathbb{R}^p_+}(H_1(x)), \Delta_{-Q}(l(x)), \Delta_{-\mathbb{R}^m_+}(H_2(x))) + \delta_{C\cap E}(x),$$

it follows that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $(x_n)_{n\in\mathbb{N}} \to \overline{x}$ and

$$\max(\Delta_{-\mathrm{Int}\mathbb{R}^p_+}(H_1(x_n)), \Delta_{-Q}(l(x_n)), \Delta_{-\mathbb{R}^m_+}(H_2(x_n))) + \delta_{C\cap E}(x_n) < 0.$$

This implies with Proposition 2.1 and relation (2), that for all $n \in \mathbb{N}$

$$x_n \in C,\ Ax_n = b,\quad H_1(x_n) \in -\mathrm{Int}\mathbb{R}^p_+,$$
$$l(x_n) \in -\mathrm{Int}Q \text{ and } H_2(x_n) \in -\mathrm{Int}\mathbb{R}^m_+$$

that is , for all n,

$$x_n \in C,\ Ax_n = b,\ l(x_n) \in -\mathrm{Int}Q,\ h(x_n) - k(x_n) \in -\mathrm{Int}\mathbb{R}^m_+$$

and

$$f(x_n) - g(x_n) - (f(\overline{x}) - g(\overline{x})) \in -\mathrm{Int}\mathbb{R}^p_+,$$

which contradicts the fact that $\overline{x}$ is a local weak minimal solution of (VP). □

Now, we can give our first main result in this section.

THEOREM 3.1. *If* $\overline{x}$ *is a local weak minimal solution of* (VP) *then for all* $(x_1^*, \ldots, x_p^*) \in \partial g_1(\overline{x}) \times \ldots \times \partial g_p(\overline{x})$ *and* $(w_1^*, \ldots, w_m^*) \in \partial k_1(\overline{x}) \times \ldots \times \partial k_m(\overline{x})$ *there exist* $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}_+^p$, $(\beta_1, \ldots, \beta_m) \in \mathbb{R}_+^m$ *and* $z^* \in (-Q)^o$ *not all zero such that* $\langle z^*, l(\overline{x})\rangle = 0$, $\beta_j(h_j(\overline{x}) - k_j(\overline{x})) = 0$, $j = 1, \ldots, m$ *and*

$$\sum_{i=1}^{p} \alpha_i x_i^* + \sum_{i=1}^{p} \beta_j w_j^* \in \partial \left(\sum_{i=1}^{p} \alpha_i f_i + \sum_{j=1}^{m} \beta_j h_j + z^* o l + \delta_{C\cap E} \right)(\overline{x}). \tag{4}$$

Proof. Put $\overline{z} := l(\overline{x})$ and $\overline{w} := h(\overline{x}) - k(\overline{x})$. By Lemma 3.1, we have

$$0 \in \partial(\max(\Delta_{-\mathrm{Int}\mathbb{R}_+^p}(H_1(\cdot)), \Delta_{-Q}(l(\cdot)), \Delta_{-\mathbb{R}_+^m}(H_2(\cdot))) + \delta_{C\cap E})(\overline{x}), \tag{5}$$

where $H_1(x) = f(x) - f(\overline{x}) - (\langle x_1^*, x - \overline{x}\rangle, \ldots, \langle x_p^*, x - \overline{x}\rangle)$ and $H_2(x) = h(x) - k(\overline{x}) - (\langle w_1^*, x - \overline{x}\rangle, \ldots, \langle w_m^*, x - \overline{x}\rangle)$. Define the functions $H : X \to \mathbb{R}^p \times Z \times \mathbb{R}^m$ and $\Phi : \mathbb{R}^p \times Z \times \mathbb{R}^m \to \mathbb{R}$ by

$$H(x) = (H_1(x), l(x), H_2(x)) \tag{6}$$

and

$$\Phi(y, z, w) = \max(\Delta_{-\mathrm{Int}\mathbb{R}_+^p}(y), \Delta_{-Q}(z), \Delta_{-\mathbb{R}_+^m}(w)). \tag{7}$$

By (6) and (7), it follows that (5) can be written as

$$0 \in \partial(\Phi o H + \delta_{C\cap E})(\overline{x}).$$

Consider now, the following functions:

$$\begin{aligned} &f_1 : X \times \mathbb{R}^p \times Z \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\} \\ &\quad (x, y, z, w) \mapsto \delta_{C\cap E}(x) + \delta_{\mathrm{epi}(H)}(x, y, z, w), && (8) \\ &f_2 : X \times \mathbb{R}^p \times Z \times \mathbb{R}^m \to \mathbb{R} \\ &\quad (x, y, z, w) \mapsto \Phi(y, z, w), && (9) \end{aligned}$$

where epigraph epi(H) of H is taken with respect to the cone $\mathbb{R}_+^p \times Q \times \mathbb{R}_+^m$. In view of Lemma 2.1, the function Φ is $(\mathbb{R}_+^p \times Q \times \mathbb{R}_+^m)$-increasing on $\mathbb{R}^p \times Z \times \mathbb{R}^m$. Hence for any $x \in X$,

$$(\Phi o H + \delta_{C\cap E})(x) = \inf_{(y,z,w)\in\mathbb{R}^p\times Z\times\mathbb{R}^m} \{f_1(x, y, z, w) + f_2(x, y, z, w)\}.$$

It is easy to see that

$$0 \in \partial(\Phi o H + \delta_{C\cap E})(\overline{x}) \text{ if and only if } (0, 0, 0, 0) \in \partial(f_1 + f_2)(\overline{x}, 0, \overline{z}, \overline{w}). \tag{10}$$

By Proposition 2.1, f_2 is convex and continuous. Since f_1 is convex and proper, it follows by the classical Moreau–Rockafellar subdifferential formula (see [15, p. 62], [16 or 17])

$$(0,0,0,0) \in \partial f_1(\overline{x},0,\overline{z},\overline{w}) + \partial f_2(\overline{x},0,\overline{z},\overline{w}). \tag{11}$$

Then there exist

$$(u_1^*, -\alpha, -z^*, -\beta) \in \partial f_1(\overline{x},0,\overline{z},\overline{w}) \text{ and } (u_2^*, \alpha_1, z_1^*, \beta_1) \in \partial f_2(\overline{x},0,\overline{z},\overline{w}) \tag{12}$$

such that

$$(0,0,0,0) = (u_1^*, -\alpha, -z^*, -\beta) + (u_2^*, \alpha_1, z_1^*, \beta_1). \tag{13}$$

We conclude from (7), (9) with (13) that (12) is equivalent to

$$(\alpha, z^*, \beta) \in \partial \Phi(0, \overline{z}, \overline{w}) \tag{14}$$

and

$$(0, -\alpha, -z^*, -\beta) \in \partial f_1(\overline{x},0,\overline{z},\overline{w}).$$

By Theorem 4.4.2, p. 267 of [10] and (14), we have

$$(\alpha, z^*, \beta) \in co(A),$$

where $A := (\partial\Delta_{-\mathrm{Int}\mathbb{R}_+^p}(0) \times \{(0,0)\}) \cup (\{0\} \times \partial\Delta_{-Q}(\overline{z}) \times \{0\}) \cup (\{(0,0)\} \times \partial\Delta_{-\mathbb{R}_+^m}(\overline{w}))$ and co is the convex hull. Hence by Theorem 3.3, p. 18 of [17], there exist $u^* \in \partial\Delta_{-\mathrm{Int}\mathbb{R}_+^p}(0)$, $v^* \in \partial\Delta_{-Q}(\overline{z})$, $w^* \in \partial\Delta_{-\mathbb{R}_+^m}(\overline{w})$ and $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_+^3$ such that

$$(\alpha, z^*, \beta) = \lambda_1(u^*, 0, 0) + \lambda_2(0, v^*, 0) + \lambda_3(0, 0, w^*) \text{ and } \lambda_1 + \lambda_2 + \lambda_3 = 1.$$

Proposition 2.2, shows that

$$(\alpha, z^*, \beta) \neq (0,0,0).$$

From definitions of $\partial\Delta_{-\mathrm{Int}\mathbb{R}_+^p}(0)$, $\partial\Delta_{-Q}(\overline{z})$ and $\partial\Delta_{-\mathbb{R}_+^m}(\overline{w})$, we immediately obtain $\alpha \in \mathbb{R}_+^p, z^* \in (-Q)^o$ and $\beta \in \mathbb{R}_+^m$. Put $\alpha := (\alpha_1, \ldots, \alpha_p)$ and $\beta := (\beta_1, \ldots, \beta_p)$. Since $\Delta_{-\mathrm{Int}\mathbb{R}_+^p}(0) = 0$, it follows by Proposition 2.1 that $\Phi(0, \overline{z}, \overline{w}) = 0$, hence by (14) we have

$$\Phi(y, z, w) \geqslant \langle \alpha, y \rangle + \langle z^*, z - \overline{z} \rangle + \langle \beta, w - \overline{w} \rangle \tag{15}$$

for all $(y, z, w) \in Y \times Z \times W$. Since $\overline{z} := l(\overline{x}) \in -Q$ and $z^* \in (-Q)^o$ we conclude from (15) with $y = 0, z = 0$ and $w = \overline{w}$

$$\langle z^*, l(\overline{x})\rangle = 0.$$

As $\overline{w} := h(\overline{x}) - k(\overline{x})) \in -\mathbb{R}^m_+$ and $\beta \in \mathbb{R}^m_+$, it follows from (15) with $y = 0, z = \overline{z}$ and $w = 0$

$$\beta_j(h_j(\overline{x}) - k_j(\overline{x})) = 0, \quad j = 1, \ldots, m.$$

Since $(0, -\alpha, -z^*, -\beta) \in \partial f_1(\overline{x}, 0, \overline{z}, \overline{w})$, it follows from (8),

$$\langle -\alpha, y\rangle + \langle -z^*, z - \overline{z}\rangle + \langle -\beta, w - \overline{w}\rangle \leqslant \delta_{C\cap E}(x) + \delta_{\mathrm{epi}(H)}(x, y, z, w) \quad (16)$$

for any $(x, y, z, w) \in X \times \mathbb{R}^p \times Z \times \mathbb{R}^m$. For each $x \in C \cap E$, setting in (16), $y = H_1(x)$, $z = l(x)$ and $w = H_2(x)$ one has

$$\sum_{i=1}^{p} \alpha_i f_i(x) + \sum_{j=1}^{m} \beta_j h_j(x) + \langle z^*, l(x)\rangle$$
$$\geqslant \sum_{i=1}^{p} \alpha_i f_i(\overline{x}) + \sum_{j=1}^{m} \beta_j h_j(\overline{x}) + \langle z^*, l(\overline{x})\rangle + \left\langle \sum_{i=1}^{p} \alpha_i x_i^* + \sum_{j=1}^{m} \beta_j w_j^*, x - \overline{x}\right\rangle. \quad (17)$$

Hence

$$\sum_{i=1}^{p} \alpha_i x_i^* + \sum_{j=1}^{m} \beta_j w_j^* \in \partial\Big(\sum_{i=1}^{p} \alpha_i f_i + \sum_{j=1}^{m} \beta_j h_j + z^* o l + \delta_{C\cap E}\Big)(\overline{x}).$$

This completes the proof of Theorem 3.1. □

As a consequence of Theorem 3.1, we have the following result

COROLLARY 3.1. *Assume that $\mathbb{R}_+(C - E)$ is a closed vector subspace, that the range of A is closed and that for $i = 1, \ldots, p$; and $j = 1, \ldots, m$, the functions f_i, h_j and l are continuous at some point $x_0 \in C \cap E$. If $\overline{x}$ is a local weak minimal solution of* (VP) *then for all $(x_1^*, \cdots, x_p^*) \in \partial g_1(\overline{x}) \times \cdots \times \partial g_p(\overline{x})$ and $(w_1^*, \ldots, w_m^*) \in \partial k_1(\overline{x}) \times \cdots \times \partial k_m(\overline{x})$ there exist $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+, (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m_+$ and $z^* \in (-Q)^o$ not all zero such that $\langle z^*, l(\overline{x})\rangle = 0$, $\beta_j(h_j(\overline{x}) - k_j(\overline{x})) = 0, j = 1, \ldots, m$ and*

$$\sum_{i=1}^{p} \alpha_i x_i^* + \sum_{j=1}^{m} \beta_j w_j^* \in \sum_{i=1}^{p} \alpha_i \partial f_i(\overline{x}) + \sum_{j=1}^{m} \beta_j \partial h_j(\overline{x}) + \partial(z^* o l)(\overline{x}) + N_C(\overline{x})$$
$$+\mathrm{rang}(A^*), \quad (18)$$

where A^* is the transpose of A and rang (A^*) is the range of A^*.

Proof. By Theorem 3.1, it suffices to show that (4) is equivalent to (18). From the classical Moreau-Rockafellar formula (see [15, p. 62], [16] or [17]) and Theorem 2.1, we conclude that (4) is equivalent to

$$\sum_{i=1}^{p}\alpha_i x_i^* + \sum_{j=1}^{m}\beta_j w_j^* \in \sum_{i=1}^{p}\alpha_i \partial f_i(\overline{x}) + \sum_{j=1}^{m}\beta_j \partial h_j(\overline{x}) + \partial(z^* ol)(\overline{x}) + N_C(\overline{x}) + N_E(\overline{x}).$$

Since the range of A is closed then by Lemma 2.4 (i) of Jeyakumar and Wolkowicz [11], we have $N_E(\overline{x}) = \text{rang}(A^*)$. This completes the proof of Corollary 3.1. □

Corollary 3.1, with $g_i = 0, i = 1, \dots, p$ and $k_j = 0, j = 1, \dots, m$ extends the result by Kanniappan [12, Theorem 3.1] and some others publications (see for instance [14]).

Let us turn to sufficient optimality conditions for (VP). Before, let us recall the following concept introduced in Hiriart-Urruty [9]. A function ψ: $X \to \mathbb{R}$ is said to be a polyhedral (or piecewise affine) convex function if $\psi(x) = \max\{\langle a_i^*, x\rangle + d_i : i = 1, \dots, q\}$ for all $x \in X$, where $a_1^*, \dots, a_q^*$ are in X^* and $d_1, \dots, d_q$ are real numbers. The proof of the following Proposition 3.1 uses some ideas of Hiriart-Urruty [9].

PROPOSITION 3.1. *Let $\overline{x} \in F$. Assume that g_i and k_j are polyhedrals, $i = 1, \dots, p$ and $j = 1, \dots, m$. Moreover, assume that for all $(x_1^*, \dots, x_p^*) \in \partial g_1(\overline{x}) \times \cdots \times \partial g_p(\overline{x})$ and $(w_1^*, \dots, w_m^*) \in \partial k_1(\overline{x}) \times \cdots \times \partial k_m(\overline{x})$ there exist $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{(0, \dots, 0)\}$, $(\beta_1, \dots, \beta_m) \in \mathbb{R}_+^m$ and $z^* \in (-Q)^o$ such that $\langle z^*, l(\overline{x})\rangle = 0, \beta_j(h_j(\overline{x}) - k_j(\overline{x})) = 0$, $j = 1, \dots, m$ and*

$$\sum_{i=1}^{p}\alpha_i x_i^* + \sum_{j=1}^{m}\beta_j w_j^* \in \partial\left(\sum_{i=1}^{p}\alpha_i f_i + \sum_{j=1}^{m}\beta_j h_j + z^* ol + \delta_{C\cap E}\right)(\overline{x}).$$

Then $\overline{x}$ is a local weak minimal solution of (VP).

Proof. For each $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, m\}$, put

$$g_i(x) := \max\{\langle a_{i,s}^*, x\rangle + a_{i,s} : s = 1, \dots, q_1\}$$

and

$$k_j(x) := \max\{\langle e_{j,s}^*, x\rangle + e_{j,s} : s = 1, \dots, q_2\}$$

for all $x \in X$, where $a_{i,1}^*, \dots, a_{i,q_1}^*$ (respectively $e_{j,1}^*, \dots, e_{j,q_2}^*$) are in X^* and $a_{i,1}, \dots, a_{i,q_1}$ (respectively $e_{j,1}, \dots, e_{j,q_2}$) are real numbers. For each $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, m\}$, we know:

$$\partial g_i(x) = \text{convex hull of } \{a^*_{i,s} : s \in I_1(x)\}$$

and

$$\partial k_j(x) = \text{convex hull of } \{e^*_{j,s} : s \in I_2(x)\}$$

for all $x \in X$, where $I_1(x)$ (respectively $I_2(x)$) denotes the set of indices $s \in \{1, \dots, q_1\}$ (respectively $s \in \{1, \dots, q_2\}$) for which $g_i(x) = \langle a^*_{i,s}, x\rangle + a_{i,s}$ (respectively $k_j(x) = \langle e^*_{j,s}, x\rangle + e_{j,s}$). By the fact that g_i and $k_j, i = 1, \dots, p, j = 1, \dots, m$ are polyhedrals, it follows from J.-B. Hiriart-Urruty [9] that for each $i \in \{1, \dots, p\}$ (respectively $j \in \{1, \dots, m\}$) there exists a neighborhood Ω_i (respectively V_j) of $\overline{x}$ such that

$$\partial g_i(x) \subset \partial g_i(\overline{x}) \text{ for all } x \in \Omega_i$$
$$\text{respectively } \partial k_j(x) \subset \partial k_j(\overline{x}) \text{ for all } x \in V_j).$$

For all $i \in \{1, \dots, p\}$ (respectively $j \in \{1, \dots, m\}$) and $x^*_i \in \partial g_i(x)$ (respectively $w^*_j \in \partial k_j(x)$), by definition, we have

$$g_i(\overline{x}) \geqslant g_i(x) + \langle x^*_i, \overline{x} - x\rangle \tag{19}$$
$$\text{(respectively } k_j(\overline{x}) \geqslant k_j(x) + \langle w^*_j, \overline{x} - x\rangle). \tag{20}$$

Set $\Omega := (\cap_{i=1}^p \Omega_i) \cap (\cap_{j=1}^m V_j)$. By our assumption, we have

$$\sum_{i=1}^p \alpha_i f_i(x) + \sum_{j=1}^m \beta_j h_j(x) + \langle z^*, l(x)\rangle \geqslant \sum_{i=1}^p \alpha_i f_i(\overline{x}) + \sum_{j=1}^m \beta_j k_j(\overline{x}) + \left\langle \sum_{i=1}^p \alpha_i x^*_i + \sum_{j=1}^m \beta_i w^*_i, x - \overline{x} \right\rangle. \tag{21}$$

for all $x \in \Omega \cap C \cap E$. Combining (19), (20) and (21) yields:

$$\sum_{i=1}^p \alpha_i (f_i(x) - g_i(x) - (f_i(\overline{x}) - g_i(\overline{x}))) + \sum_{i=1}^m \beta_j (h_j(x) - k_j(x)) + \langle z^*, l(x)\rangle \geqslant 0$$

for all $x \in \Omega \cap C \cap E$. In particular, with relation (1), it follows

$$\sum_{i=1}^p \alpha_i (f_i(x) - g_i(x) - (f_i(\overline{x}) - g_i(\overline{x}))) \geqslant 0$$

for all $x \in \Omega \cap C \cap E \cap D$. By the fact that $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+ \setminus \{(0, \ldots, 0)\}$, it follows that $\bar{x}$ is a local weak minimal solution of (VP). The proof of Proposition 3.1 is completed. □

We will need the following Proposition 3.2 which will be used in Corollary 3.2.

PROPOSITION 3.2. *Let $\bar{x} \in F$. Assume that there exists $x_0 \in X$ such that $x_0 \in C$, $Ax_0 = b$, $l(x_0) \in -\mathrm{Int}Q$ and $h_j(x_0) - k_j(x_0) \leqslant 0$, $j = 1, \ldots, m$. If for each $(x_1^*, \ldots, x_p^*) \in \partial g_1(\bar{x}) \times \cdots \times \partial g_p(\bar{x})$ and $(w_1^*, \ldots, w_m^*) \in \partial k_1(\bar{x}) \times \cdots \times \partial k_m(\bar{x})$ there exist $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+$, $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m_+$ and $z^* \in (-Q)^o$ not all zero such that $\langle z^*, l(\bar{x})\rangle = 0$, $\beta_j(h_j(\bar{x}) - k_j(\bar{x})) = 0$, $j = 1, \ldots, m$ and*

$$\sum_{i=1}^{p} \alpha_i x_i^* + \sum_{j=1}^{p} \beta_j w_j^* \in \partial \left(\sum_{i=1}^{p} \alpha_i f_i + \sum_{j=1}^{m} \beta_j h_j + z^* o l + \delta_{C \cap E} \right)(\bar{x}).$$

Then $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_m) \neq (0, \ldots, 0, 0, \ldots, 0)$.

Proof. Suppose that $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_m) = (0, \ldots, 0, 0, \ldots, 0)$. Hence for all $x \in C \cap E$

$$\langle z^*, l(x)\rangle \geqslant 0. \tag{22}$$

Since $l(x_0) \in -\mathrm{Int}Q$ then there exists a neighborhood V of zero in Z such that

$$l(x_0) + V \subset -Q.$$

This implies by relation (22)

$$\langle z^*, z\rangle \leqslant 0 \quad \text{for all } z \in V.$$

Hence $z^* = 0$, which contradicts the fact that $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_m, z^*) \neq (0, \ldots, 0, 0, \ldots, 0, 0)$. This implies that $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_m) \neq (0, \ldots, 0, 0, \ldots, 0)$. The proof is completed. □

In the following, we derive necessary optimality conditions for (VP) in terms of Lagrange-Karush-Kuhn-Tucker multipliers whenever, the set of constraints is convex. More precisely, we consider the following vector mathematical programming:

$$(\mathrm{VPI}) \begin{cases} \mathbb{R}^p_+ - \text{ Minimize } f(x) - g(x), \\ \quad \text{subject to the constraints} \\ \quad x \in C, l(x) \in -Q \text{ and } Ax = b, \end{cases}$$

where f, g, C, l, A and Q are as in (VP).

COROLLARY 3.2. *Assume that, there exists* $x_0 \in X$ *such that* $x_0 \in C \cap \{x \in X : Ax_0 = b, l(x) \in -\mathrm{Int}Q\}$ (*this condition is often said to be the generalized Slater's condition see Jeyakumar and Wolkowicz* [*11*]). *If* $\overline{x}$ *is a local weak minimal solution of* (VPI) *then for each* $(x_1^*, \dots, x_p^*) \in \partial g_1(\overline{x}) \times \cdots \times \partial g_p(\overline{x})$ *there exist* $\alpha \in \mathbb{R}_+^p \setminus \{0\}$ *with* $\alpha = (\alpha_1, \dots, \alpha_p)$ *and* $z^* \in (-Q)^o$ *such that* $\langle z^*, l(\overline{x})\rangle = 0$ *and*

$$\sum_{i=1}^p \alpha_i x_i^* \in \partial \left(\sum_{i=1}^p \alpha_i f_i + z^* o l + \delta_{C\cap E} \right)(\overline{x}).$$

Proof. The proof follows from Theorem 3.1 and Proposition 3.2. □

Let us turn to necessary and sufficient optimality conditions for (VPI) in terms of Lagrange-Karush-Kuhn-Tucker multipliers.

COROLLARY 3.3. *Let* $\overline{x} \in C$ *with* $l(\overline{x}) \in -Q$ *and* $A\overline{x} = b$. *Assume that* g_i, $i = 1, \dots, p$ *are polyhedrals and that, there exists* $x_0 \in X$ *such that*

$$x_0 \in C \cap \{x \in X : h(x) \in -\mathrm{Int}Q \text{ and } Ax_0 = b\}.$$

Then $\overline{x}$ *is a local weak minimal solution of* (VPI) *if and only if for each* $(x_1^*, \dots, x_2^*) \in \partial g_1(\overline{x}) \times \cdots \times \partial g_p(\overline{x})$ *there exist* $\alpha \in \mathbb{R}_+^p \setminus \{0\}$ *with* $\alpha = (\alpha_1, \dots, \alpha_p)$ *and* $z^* \in (-Q)^o$ *such that* $\langle z^*, l(\overline{x})\rangle = 0$ *and*

$$\sum_{i=1}^p \alpha_i x_i^* \in \partial \left(\sum_{i=1}^p \alpha_i f_i + z^* o l + \delta_{C\cap E} \right)(\overline{x}).$$

Proof. The proof follows from Corollary 3.2 and Proposition 3.1. □

Corollary 3.3, with $g_i = 0, i = 1, \dots, p$ and with the assumptions as in Corollary 3.1, extends the result by Minami [14, Theorem 5.1].

In the following Corollary 3.4, we derive from Theorem 3.1, necessary optimality conditions for convex vector maximization problems. Consider the following convex vector maximization problem

$$\text{(VPII)} \begin{cases} \mathbb{R}_+^P - \text{ Maximize } g(x) = (g_1(x), \dots, g_p(x)) \\ \quad\quad \text{subject to } x \in C. \end{cases}$$

Remark 3.1. It is easy to see that $\overline{x}$ is a local weak maximal solution of (VPII) if and only if $\overline{x}$ is a local weak minimal solution of the problem (VPII′) defined by

$$\text{(VPII}')\begin{cases}\mathbb{R}^p_+-\text{ Minimize } -g(x)=(-g_1(x),\dots,-g_p(x))\\ \qquad\text{subject to } x\in C.\end{cases}$$

As a consequence of Theorem 3.1 we have the following result.

COROLLARY 3.4. *If* $\overline{x}$ *is a local weak maximal solution of* (VPII), *then for all* $(x_1^*,\dots,x_p^*)\in\partial g_1(\overline{x})\times\cdots\times\partial g_p(\overline{x})$ *there exists* $\alpha\in\mathbb{R}^p_+\backslash\{0\}$ *with* $\alpha=(\alpha_1,\dots,\alpha_p)$ *such that*

$$\sum_{i=1}^p \alpha_i x_i^*\subset N_C(\overline{x}).$$

Proof. The proof follows from Remark 3.1 and Theorem 3.1. □

Corollary 3.4, extends the result obtained in scalar optimization by Hiriart-Urruty [9, Proposition 3.8] to the vector case.

4. Application to Vector Fractional Mathematical Programming

In this section, we give an application. Let $f_i\colon X\to\mathbb{R}_+$, $g_i: X\to\mathbb{R}_+\setminus\{0\}$; be given functionals which are convex, $i=1,\dots,p$. Under these assumptions we investigate the vector optimization problem

$$(VFP)\begin{cases}\mathbb{R}^p_+-\text{ Minimize } \left(\frac{f_1(x)}{g_1(x)},\frac{f_2(x)}{g_2(x)},\dots,\frac{f_p(x)}{g_p(x)}\right)\\ \qquad\text{subject to the constraints}\\ \qquad x\in C, Ax=b, l(x)\in -Q \text{ and}\\ \qquad h_j(x)-k_j(x)\leqslant 0 \text{ for } j=1,\dots,m,\end{cases}$$

where C, A, l, h_j and k_j are as in problem (VP).

We will need the following lemma.

LEMMA 4.1. $\overline{x}$ *is a local weak minimal solution of* (VFP) *if and only if* $\overline{x}$ *is a local weak minimal solution of the following problem*

$$(VFP')\begin{cases}\mathbb{R}^p_+-\text{ Minimize } (f_1(x)-r_1g_1(x), f_2(x)-r_2g_2(x),\dots,\\ \qquad\qquad f_p(x)-r_pg_p(x))\\ \qquad\text{subject to the constraints}\\ \qquad x\in C,\ Ax=b,\ l(x)\in -Q \text{ and}\\ \qquad h_j(x)-k_j(x)\leqslant 0 \text{ for } j=1,\dots,m,\end{cases}$$

where $r_i=f_i(\overline{x})/g_i(\overline{x})$, $i=1,2,\dots,p$.

Proof. The only if part. Suppose the contrary. There exists $(x_n)_{n\in\mathbb{N}} \subset X$ and an integer n_0 such that $(x_n)_{n\in\mathbb{N}} \to \overline{x}$ and for all $n \geqslant n_0$

$$h_j(x_n) - k_j(x_n) \leqslant 0,\ \ j = 1, 2, \dots, m;\ Ax_n = b,\ l(x_n) \in -Q,$$

and

$$(f_i(x_n) - r_i g_i(x_n) - (f_i(\overline{x}) - r_i g_i(\overline{x})) < 0, \quad i = 1, 2, \dots, p.$$

Due to the fact that for $i = 1, 2, \dots, p,\ g_i(x_n) > 0$, it follows

$$\frac{f_i(x_n)}{g_i(x_n)} - \frac{f_i(\overline{x})}{g_i(\overline{x})} < 0, \quad i = 1, 2, \dots, p,$$

which contradicts the fact that $\overline{x}$ is a local weak minimal solution of (VFP). So $\overline{x}$ is a local weak minimal solution of (VFP$'$).

The converse implication can be proved in the similar way. The proof is completed. □

THEOREM 4.1. *Assume that l, f_i and h_j for $i = 1, \dots, p$ and $j = 1, \dots, m$ are continuous at some point x_0 of $C \cap E$, that $\mathbb{R}_+(C - E)$ is a closed vector subspace and that the range of A is closed. If $\overline{x}$ is a local weak minimal solution of (VFP)* then for each $(x_1^*, \dots, x_p^*) \in \partial g_1(\overline{x}) \times \cdots \times \partial g_p(\overline{x})$ *and* $(w_1^*, \dots, w_m^*) \in \partial k_1(\overline{x}) \times \cdots \times \partial k_m(\overline{x})$ *there exist* $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p$, $(\beta_1, \dots, \beta_m) \in \mathbb{R}_+^m$ *and* $z^* \in (-Q)^o$ *not all zero and* $w^* \in W^*$ *such that* $\langle z^*, l(\overline{x})\rangle = 0$, $\beta_j(h_j(\overline{x}) - k_j(\overline{x})) = 0,\ j = 1, \dots, m$ *and*

$$\sum_{i=1}^{p} \alpha_i r_i x_i^* + \sum_{j=1}^{m} \beta_j w_j^* \in \sum_{i=1}^{p} \alpha_i \partial f_i(\overline{x}) + \sum_{j=1}^{m} \beta_j \partial h_j(\overline{x}) + \partial(z^* o l)(\overline{x}) + A^* w^* + N_C(\overline{x}),$$

where $r_i = f_i(\overline{x})/g_i(\overline{x}),\ i = 1, 2, \dots, p$.

Proof. The proof follows from Lemma 4.1 and Corollary 3.1. □

Acknowledgements

The author would like to thank the anonymous referees for their valuables comments and suggestions.

References

1. Amahroq, T. and Taa, A. (1997), On Lagrange–Kuhn–Tucker multipliers for multiobjective optimization problems, 41, 159–172.
2. Attouch, H. and Brezis, H. (1986), Duality for the Sum of Convex Functions in General Banach Spaces. In: Barroso, J.A. (ed.), *Aspects of Mathematics and its Application*, Elsevier Science Publishers B.V., Amsterdam, pp. 125–133.
3. Censor, Y. (1977), Pareto optimality in multiobjective problems, *Applied Mathematics and Optimization*, 4, 41–59.
4. Ciligot-Travain, M. (1994), On Lagrange–Kuhn–Tucker multipliers for Pareto optimization problems, *Numer. Funct. Anal. Optim.*, 15, 689–693.
5. Charnes, A., Cooper, W.W., Huang, Z.M. and Wei, Q.L. (1990), Fundamental theorems of nondominated solutions associated with cones in normed linear spaces, *Journal of Mathematical Analysis and Applications*, 150, 54–78.
6. Charnes, A., Huang, Z.M., Rousseau, J.J. and Wei, Q.L. (1990), Cone extremal solutions of multi-payoff games with cross-constrained strategy sets, *Optimization*, 21, 51–69.
7. Dauer, J.P. and Saleh, O.A. (1993), A characterization of proper minimal points as solutions of sublinear optimization problems, *J. Mathematical Analysis and Applications*, 178, 227–246.
8. Hiriart-Urruty, J.B. (1979), Tangent cones, generalized gradients and mathematical programming in banach spaces, *Math. Ope. Res.*, 4, 79–97.
9. Hiriart-Urruty, J.B. (1989), From Convex Optimization to Nonconvex Optimization. In: Clarke, F.H., Demyanov, V.F. and Giannessi, F. (eds.), *Nonsmooth Optimization and Related Topics*, Plenum Press, pp. 219–239.
10. Hiriart-Urruty, J.B. and Lemaréchal, C. (1993), *Convex Analysis and Minimization Algo rithms I*, Springer Verlag, Berlin, Germany.
11. Jeyakumar, V. and Wolkowicz, H. (1992), Generalizations of Slater's constraint qualification for infinite convex programs, *Mathematical Programming*, 57, 85–101.
12. Kanniappan, P. (1983), Necessary conditions for optimality of nondifferentiable convex multiobjective programming, *Journal of Optimization Theory and Applications*, 40(2), 167–173.
13. Li, S.X. (1996), Quasiconvexity and nondominated solutions in multiobjective programming, *Journal of Optimization Theory and Applications*, 88, 1997–208.
14. Minami, M. (1981), Weak Pareto optimality of multiobjective problems in locally convex linear topological space, *Journal of Optimization Theory and Applications*, 34(4), 469–484.
15. Moreau, J.-J. (1967), Fonctionelles Convexes, Mimeographed Lecture Notes, Collège de France.
16. Rockafellar, R.T. (1974), Conjugate duality and optimization, *Conference Board of Mathematical Sciences Series* 16, SIAM, Publications, Philadelphia.
17. Rockafellar, R.T. (1970), *Convex Analysis*, Princeton University Press
18. Rodriguez, B. and Simons, S. (1988), Conjugate functions and subdifferentials in normed situations for operators with complete graphs, *Nonlinear Analysis*, 12, 1069–1078.
19. Swartz, C. (1987), Pschenichny's theorem for vector optimization, *Journal of Optimization Theory and Appli.*, 53, 309–317.
20. Taa, A. (1999), Optimality conditions for vector mathematical programming via a theorem of the alternative, *Journal of Mathematical Analysis and Applications*, 233, 233–245.
21. Yu, P.L. (1985), *Multiple-Criteria Decisions Making: Concepts, Techniques, and Extensions*, Plenum, New York.